

Optimal Choices of Reference for Quasi-local Energy

Chiang-Mei Chen^{1,3,*}, Jian-Liang Liu^{1,3,†}, James M. Nester^{1,2,3,‡} and Ming-Fan Wu^{1,3§}

¹*Department of Physics, National Central University, Chungli 320, Taiwan*

²*Graduate Institute of Astronomy, National Central University, Chungli 320, Taiwan and*

³*Center for Mathematics and Theoretical Physics,
National Central University, Chungli 320, Taiwan*

(Dated: September 15, 2009)

We have proposed a program for determining the reference for the quasi-local energy defined in the covariant Hamiltonian formalism. Our program has been tested by applying it to the spherically symmetric spacetimes. With respect to different observers we found that the quasi-local energy can be positive, zero, or even negative. The observer measuring the maximum energy was identified; the associated energy is positive for both the Schwarzschild and the Friedmann-Lemaître-Robertson-Walker spacetimes.

PACS numbers: 04.20.Cv, 04.20.Fy, 98.80.Jk

Introduction. An outstanding fundamental problem in general relativity is that there is no proper definition for the energy density of gravitating systems. (This can be understood as a consequence of the equivalence principle.) The modern concept is that gravitational energy should be non-local, more precisely *quasi-local*, i.e., it should be associated with a closed two-surface (for a comprehensive review see [1]). Here we consider one proposal based on the covariant Hamiltonian formalism [2] wherein the quasi-local energy is determined by the Hamiltonian boundary term. For a specific spacetime displacement vector field on the boundary of a region (which can be associated with the observer), the associated quasi-local energy depends not only on the dynamical values of the fields on the boundary but also on the choice of reference values for these fields. Thus a principal issue in this formalism is the proper choice of reference spacetime for a given observer.

It is generally accepted that the gravitational energy for an asymptotically flat system should be non-negative and should vanish only for Minkowski space (see, e.g., [3]; for proofs of this property for GR see [4, 5]). In view of this it has been natural to regard these properties as desirable for a good quasi-local energy [1, 6]. The idea that a suitable reference should be the one which gives the minimal energy then followed quite naturally. We have proposed using this approach to choose the optimal reference for the covariant Hamiltonian boundary term. Here we consider this optimal choice program for both static and dynamic spherically symmetric spacetimes (the most important test cases). Specifically, for the Schwarzschild and the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes, we found the resultant quasi-local energies can be positive, zero, or even negative for different observers. However, for both cases, there is one observer who would measure the maximum energy, and for this observer the associated energy is positive. Furthermore we find that this energy-extremization program (at least for these spherically symmetric systems) is equivalent

to matching the geometry at the two-sphere boundary, which provides for a simple interpretation of the displacement vector.

The Hamiltonian Formulation. We begin with a brief review of the covariant Hamiltonian formalism [7, 8, 9, 10]. A *first order Lagrangian 4-form* for a k -form field φ can be expressed as $\mathcal{L} = d\varphi \wedge p - \Lambda(\varphi, p)$. The action should be invariant under local diffeomorphisms, which infinitesimally correspond to a displacement along some vector field N . From Noether's theorem there is a conserved translational current which can be written as a 3-form linear in the displacement vector plus a total differential: $\mathcal{H}(N) := \mathcal{L}_N \varphi \wedge p - i_N \mathcal{L} := N^\mu \mathcal{H}_\mu + d\mathcal{B}(N)$. Here $\mathcal{H}_\mu \equiv -i_\mu \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \varsigma \frac{\delta \mathcal{L}}{\delta p} \wedge i_\mu p$ with $\varsigma := (-1)^k$; this identity is a necessary consequence of *local* diffeomorphism invariance (i.e., symmetry for non-constant N^μ). Consequently \mathcal{H}_μ vanishes on shell; hence the value of the Hamiltonian—the conserved quantity associated with a local displacement N and a spatial region Σ —is determined by a 2-surface integral over the region's boundary:

$$E(N, \Sigma) := \int_\Sigma \mathcal{H}(N) = \oint_{\partial\Sigma} \mathcal{B}(N). \quad (1)$$

For *any* choice of N this expression defines a conserved quasi-local quantity. Different choices of boundary term correspond to different boundary conditions.

Einstein's gravity theory, general relativity (GR), can be formulated in several ways. For our purposes the most convenient is to take the *orthonormal coframe* $\vartheta^\mu = \vartheta^\mu_k dx^k$ and the *connection one-form* $\omega^\alpha_\beta = \Gamma^\alpha_{\beta k} dx^k$ as the geometric potentials. Moreover we take the connection to be *a priori* metric compatible: $Dg_{\alpha\beta} := dg_{\alpha\beta} - \omega^\gamma_\alpha g_{\gamma\beta} - \omega^\gamma_\beta g_{\alpha\gamma} \equiv 0$. Restricted to orthonormal frames where the metric components are constant, this condition reduces to the algebraic constraint $\omega^{\alpha\beta} \equiv \omega^{[\alpha\beta]}$.

We consider the vacuum (source free) case for simplicity. GR can be obtained from the first order Lagrangian 4-form $\mathcal{L}_{\text{GR}} = \Omega^{\alpha\beta} \wedge \rho_{\alpha\beta} + D\vartheta^\mu \wedge \tau_\mu - V^{\alpha\beta} \wedge (\rho_{\alpha\beta} - \frac{1}{2\kappa} \eta_{\alpha\beta})$, where $\Omega^\alpha_\beta := d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta$ is the *curvature 2-*

form, $D\vartheta^\mu := d\vartheta^\mu + \omega^\mu{}_\nu \wedge \vartheta^\nu$ is the *torsion* 2-form, and $\eta^{\alpha\beta\cdots} := \star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \cdots)$ is the dual form basis. The 2-forms $\Omega^{\alpha\beta}$, $V^{\alpha\beta}$ and $\rho_{\alpha\beta}$ are antisymmetric. We take $\kappa := 8\pi G/c^4 = 8\pi$. In [11] a “preferred boundary term” for GR was identified:

$$\mathcal{B}(N) = \frac{1}{16\pi} (\Delta\omega^\alpha{}_\beta \wedge \iota_N \eta^{\alpha\beta} + \bar{D}_\beta N^\alpha \Delta\eta_{\alpha}{}^\beta), \quad (2)$$

where Δ indicates the difference between the dynamic and reference values and \bar{D}_β is the reference covariant derivative. The reference values can be determined by pullback from an embedding of the boundary into a suitable reference space.

The Energy-Extremization Program. Here we explicitly formulate the extremization program for static spherically symmetric spacetimes. The Schwarzschild-like metric in “standard” spherical coordinates is given by $ds^2 = -Adt^2 + A^{-1}dr^2 + r^2 d\Omega_2^2$, where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$. However, there are other favorable coordinate choices for the Schwarzschild metric (e.g., Painlevé-Gullstrand, Eddington-Finkelstein, Kruskal-Szekeres). In order to accommodate most well-known coordinates, we consider a more general version of the Schwarzschild metric via a coordinate transformation $t = t(u, v)$, $r = r(u, v)$; the metric becomes $ds^2 = -(At_u^2 - A^{-1}r_u^2)du^2 + 2(A^{-1}r_u r_v - At_u t_v)dudv + (A^{-1}r_v^2 - At_v^2)dv^2 + r^2 d\Omega_2^2$. The Minkowski spacetime $d\bar{s}^2 = -dT^2 + dR^2 + R^2 d\Theta^2 + R^2 \sin^2\Theta d\Phi^2$ is a natural choice for the reference. However, the essential issue of the reference choice is the identification between the reference and physical spacetime coordinates. A legitimate approach is to assume $T = T(u, v)$, $R = R(u, v)$, $\Theta = \theta$, $\Phi = \varphi$ and isometrically embed the two-sphere boundary S at (t_0, r_0) and its neighborhood into the Minkowski reference such that $R_0 := R(t_0, r_0) = r_0$. Assume that the displacement vector $N = N^u \partial_u + N^v \partial_v = N^t \partial_t + N^r \partial_r = N^T \partial_T + N^R \partial_R$ is future timelike and the orientation is preserved under diffeomorphisms, i.e., $\sqrt{-\alpha} := t_u r_v - t_v r_u > 0$ and $X^{-1} := T_u R_v - T_v R_u > 0$. The second term of Eq. (2) vanishes for spherically symmetric spacetimes; the energy can then be evaluated:

$$E = \frac{r}{2} (N^u B + N^v C) \sqrt{-\alpha}, \quad (3)$$

$$B = XT_u + g^{vu}(R_v - 2r_u) + g^{vv}(R_v - 2r_v), \quad (4)$$

$$C = XT_v + g^{uu}(2r_u - R_u) + g^{uv}(2r_v - R_v), \quad (5)$$

where the subscripts indicate partial differentiations. Note that the quasi-local energy is evaluated on the boundary two-sphere S ; the variables appearing in Eq. (3) and in the following are also evaluated on S . Each choice of the embedding variables $\{T_u, T_v, R_u, R_v\}$ means a different embedding, hence a different reference. For any given displacement vector we extremize the energy with respect to the embedding variables; we get four equations, but only three are independent:

$$N^u R_u + N^v R_v = N^R = 0, \quad (6)$$

$$X^2 T_v (N^u T_u + N^v T_v) - \alpha^{-1} (g_{uv} N^u + g_{vv} N^v) = 0, \quad (7)$$

$$X^2 T_u (N^u T_u + N^v T_v) - \alpha^{-1} (g_{uu} N^u + g_{uv} N^v) = 0. \quad (8)$$

A useful combination $(8) \times R_v - (7) \times R_u$ gives

$$X(N^u T_u + N^v T_v) + \alpha^{-1} [(g_{uv} N^u + g_{vv} N^v) R_u - (g_{uu} N^u + g_{uv} N^v) R_v] = 0. \quad (9)$$

From Eq. (6) we get $R_u = -\frac{N^v}{N^u} R_v$ and $N^T := N^u T_u + N^v T_v = \frac{N^u}{X R_v}$; then R_v can be found from Eq. (9):

$$\frac{N^u}{R_v} - \alpha^{-1} \frac{R_v}{N^u} g(N, N) = 0 \quad \Rightarrow \quad R_v^2 = \frac{\alpha (N^u)^2}{g(N, N)}. \quad (10)$$

We require the displacement vector to be future timelike, i.e., $N^T > 0$ and $N^u > 0$, and the orientation to be preserved under diffeomorphisms, i.e., the Jacobians are positive. Then R_v should be positive, and therefore

$$R_v = \sqrt{\frac{\alpha}{g(N, N)}} N^u, \quad R_u = -\sqrt{\frac{\alpha}{g(N, N)}} N^v. \quad (11)$$

Now we calculate the energy. Using Eq. (11) we get

$$\sqrt{-\alpha} (N^u B + N^v C) = 2 \left(\sqrt{-g(N, N)} - AN^t \right), \quad (12)$$

where the explicit metric is used in the calculation. Choose N to be unit timelike on the two-sphere, i.e., $-1 = g(N, N) = g_{uu} (N^u)^2 + 2g_{uv} N^u N^v + g_{vv} (N^v)^2$, then the quasi-local energy for any given future timelike displacement vector N reduces to

$$E = r (1 - AN^t), \quad (13)$$

which is independent of the coordinate system. The energy expression was obtained without knowing the explicit expressions for the variables T_u, T_v . Indeed, we cannot solve for all four embedding variables, since there are only three independent equations from the extremization. Thus, this optimal program produces a unique energy with an equivalent class of references for any given physical observer. However, it is also reasonable to impose the normalization condition of the displacement vector in the reference spacetime, i.e., $\bar{g}(N, N) = -1$. Using this condition and Eqs. (7, 8, 11) we find $T_u = At_u N^t - A^{-1} r_u N^r$ and $T_v = At_v N^t - A^{-1} r_v N^r$. Then the reference is uniquely determined.

Now we can vary the energy with respect to the displacement vector to determine the observer who measures the extreme energy. In view of the constraint $g(N, N) = -A(N^t)^2 + A^{-1}(N^r)^2 = -1$ we take $\sqrt{A} N^t = \cosh z$ and $N^r/\sqrt{A} = \sinh z$, and then the energy extremization $\partial E/\partial z = 0$ implies $\cosh z = 1$ and $\frac{\partial^2 E}{\partial z^2} \big|_{\frac{\partial E}{\partial z}=0} \leq 0$. This result means that among all physical observers the *static observer*, i.e., $N = A^{-1/2} \partial_t$, would measure the *maximum* quasi-local energy

$$E = r (1 - \sqrt{A}). \quad (14)$$

For Schwarzschild this is a standard value which has been obtained by many researchers, e.g., [6, 7, 8, 12, 13].

The energy expression (14) can also be obtained via a one step approach of extremizing the energy with respect to the embedding variables and the displacement vector. We firstly require the displacement vector to be both the unit timelike Killing vector of the Minkowski spacetime

$$N = \partial_T \implies N^u = X R_v, \quad N^v = -X R_u, \quad (15)$$

and a unit timelike vector in the physical spacetime, $-1 = g(N, N)$, implying

$$X^2 = -(g_{uu}R_v^2 - 2g_{uv}R_uR_v + g_{vv}R_u^2)^{-1}. \quad (16)$$

With these assumptions, the energy (3) reduces to

$$E = \frac{r}{2} X \left[1 - \alpha^{-1} X^{-2} + 2\alpha^{-1} (g_{uv}r_u R_v - g_{uu}r_v R_v - g_{vv}r_u R_u + g_{uv}r_v R_u) \right] \sqrt{-\alpha}. \quad (17)$$

Now there are only two embedding variables appearing in the energy expression. Varying the above energy expression with respect to these two variables we get

$$(g_{vv}R_u - g_{uv}R_v)(1 + \alpha^{-1}X^{-2}) + 2R_v(r_u R_v - r_v R_u) = 0, \quad (18)$$

$$(g_{uu}R_v - g_{uv}R_u)(1 + \alpha^{-1}X^{-2}) - 2R_u(r_u R_v - r_v R_u) = 0. \quad (19)$$

The combination (18) $\times R_u +$ (19) $\times R_v$ gives $X^{-1} = \sqrt{-\alpha}$; then both (18) and (19) reduce to the condition $r_u R_v - r_v R_u = 0$. Together with Eq. (16) we get $R_u^2 = r_u^2/A$ and $R_v^2 = r_v^2/A$. We should pick the plus sign for R_u and R_v . The unique energy produced by this program is then $E = r(1 - \sqrt{A})$, which agrees with what we found above.

Energy Measured by Various Observers. For the Schwarzschild spacetime $A = 1 - 2m/r$; it is obvious that Eq. (14) can only be valid outside the black hole horizon: there is no static observer inside the black hole. In order to discuss the energy inside a black hole, let us examine our energy formula Eq. (13) for a radial geodesic observer in the Schwarzschild spacetime. For an observer who falls initially with velocity v_0 from a constant distance $r = a$, there are two different types of orbits: (1) the crash orbit—where ingoing observers crash directly into the singularity at $r = 0$, while outgoing observers first shoot out to the turning point r_{\max} and then fall back and crash into the singularity; (2) the crash/escape orbit—where ingoing observers crash, but outgoing observers can escape to infinity by having a large enough initial velocity. The displacement vector for such observers is the unit tangent of the geodesic; then $N^t = \frac{1}{1-2m/r} \sqrt{\frac{1-2m/a}{1-v_0^2}}$, where $2m < a$ and $0 < r \leq r_{\max}$. The energy measured by this observer is $E = r \left(1 - \sqrt{\frac{1-2m/a}{1-v_0^2}} \right)$. This

result agrees with that of the Brown-York quasi-local energy expression [14, 15]. One can see that the energy decreases as the initial velocity v_0 increases. When the initial velocity v_0 is less, equal, or greater than $\sqrt{2m/a}$ (which is the escape velocity from the Newtonian point of view) the energy is positive, zero, or negative, respectively. The negative value for the energy may appear odd, but it can be explained physically. It is correlated with the geometric property that the scalar curvature of the spacelike hypersurface orthogonal to the displacement vector is (unlike the usual cases) negative. Note that the ingoing geodesic observers can measure energy inside the black hole. This energy is proportional to the radial distance r , so it is a smooth function in the region $0 \leq r \leq r_{\max}$.

In addition to the static and radial geodesic observers there are other natural choices—in particular the unit normal of the constant coordinate time hypersurface in various coordinate systems. Using this idea in ingoing Eddington-Finkelstein coordinates, $ds^2 = -\frac{du^2}{1+2m/r} + \left[\frac{2mdu}{r\sqrt{1+2m/r}} + \sqrt{1+2m/r}dv \right]^2 + r^2 d\Omega_2^2$, where $du = dt + \frac{2m}{r} A^{-1} dr$, $dv = dr$, gives $E_{\text{EF}} = r[1 - (1 + 2m/r)^{-1/2}] = 2m(1 + 2m/r + \sqrt{1+2m/r})^{-1}$. Whereas for ingoing Painlevé-Gullstrand coordinates, $ds^2 = -du^2 + (\sqrt{2m/r}du + dv)^2 + r^2 d\Omega_2^2$, where $du = dt + A^{-1} \sqrt{2m/r} dr$, $dv = dr$, the energy *vanishes*. All these outcomes are smoothly dependent on r .

The Geometrical Meaning. The proposed optimal program can be interpreted as an adapted coordinate choice for any given displacement vector, i.e., observer. The adapted coordinates, denoted as $\{u', v'\}$, have the associated coframes

$$\vartheta^0 = a_u^0 du', \quad \vartheta^1 = a_u^1 du' + a_v^1 dv', \quad (20)$$

and the displacement vector is the unit timelike vector, namely $N = e_0$; its components can be expressed in terms of (a_u^0, a_u^1, a_v^1) . The embedding variables are determined by requiring on S

$$\begin{aligned} \vartheta^0 &= a_u^0 du' = dT = T_{u'} du' + T_{v'} dv', \\ \vartheta^1 &= a_u^1 du' + a_v^1 dv' = dR = R_{u'} du' + R_{v'} dv'. \end{aligned} \quad (21)$$

This identification leads directly to (13). Thus, the extremization program, in these adapted coordinates, yields an isometric embedding on the two-sphere boundary in a way such that N is the unit timelike vector in both the physical and Minkowski reference spacetimes.

Dynamic Spherically Symmetric Spacetimes. The optimal reference choice program can likewise be applied to dynamic spherically symmetric spacetimes, the FLRW cosmological models. With a calculation similar to that above, we found that the optimal energy for a given dis-

placement vector is

$$E = ar \left(1 - \sqrt{1 - kr^2} N^t - \frac{a\dot{a}r}{\sqrt{1 - kr^2}} N^r \right), \quad (22)$$

where $\dot{a} = da/dt$ and $k = -1, 0, 1$ is the sign of the spatial curvature. An obvious choice is the *comoving* observer, $N = \partial_t$, then the energy is $E = ar(1 - \sqrt{1 - kr^2})$ —a value which has been found previously [16]—which is *negative* for negative curvature. On the other hand, varying the energy (22) with respect to the displacement vector we find the maximum energy

$$E = \frac{ar^3(k + \dot{a}^2)}{1 + \sqrt{1 - kr^2 - \dot{a}^2 r^2}}, \quad (23)$$

(which again does not exist for all observer locations r). The associated displacement vector is

$$N = \frac{\sqrt{1 - kr^2}}{\sqrt{1 - kr^2 - \dot{a}^2 r^2}} \partial_t - \frac{\dot{a}r}{a} \frac{\sqrt{1 - kr^2}}{\sqrt{1 - kr^2 - \dot{a}^2 r^2}} \partial_r, \quad (24)$$

which is just the unit dual mean curvature vector [17]. For a dust model with matter energy density ρ , by imposing the Friedmann equation, $k + \dot{a}^2 = (8\pi/3)\rho a^2$, the energy expression (23) becomes

$$E = \frac{(8\pi/3)\rho a^3 r^3}{1 + \sqrt{1 - (8\pi/3)\rho a^2 r^2}} \geq 0. \quad (25)$$

For small r the proper matter interior limit required by the equivalence principle is evident. While we find that the observer dependent quasi-local energy can be *negative*, for the particular observer who measures the maximum value, the quasi-local energy is *positive*.

Conclusion. The covariant Hamiltonian quasi-local energy expression (2) has certain nice properties, [7, 8, 9, 10, 11], but it suffers from two ambiguities: which displacement vector and which reference? We have proposed isometrically embedding the two-sphere boundary and its neighborhood in the dynamic spacetime into the Minkowski reference and then extremizing the energy to determine the embedding variables. Here we have discussed the application of this program to static and dynamic spherically symmetric spacetimes where we obtain, for any given future timelike displacement, a unique quasi-local energy which can be positive, zero, or even negative. When we further vary the energy with respect to the displacement vector we can identify a special observer who measures the maximum—and always positive—energy.

Moreover, the optimal program is, at least in this spherical case, actually equivalent to an adapted coordinate choice for each observer. In such coordinates,

our program not only isometrically matches the geometry near the two-sphere boundary, but also identifies the displacement vector as the unit timelike vector orthogonal to the constant coordinate time hypersurface, and, at the same time, as the unit timelike Killing vector of the Minkowski reference. This observation lends further support to the proposed optimal program.

We believe that this spherical case is the main test case; it shows that our program has promise as a universal approach for determining the reference needed for the covariant Hamiltonian boundary term quasi-local energy for general spacetimes.

This work was supported by the National Science Council of the R.O.C. under the grants NSC-98-2112-M-008-008 (JMN) and NSC 96-2112-M-008-006-MY3 (CMC) and in part by the National Center of Theoretical Sciences (NCTS).

* Electronic address: cmchen@phy.ncu.edu.tw

† Electronic address: tendauliang@gmail.com

‡ Electronic address: nester@phy.ncu.edu.tw

§ Electronic address: 93222036@cc.ncu.edu.tw

- [1] L. B. Szabados, Living Rev. Rel. **12**, 4 (2009).
- [2] J. M. Nester, Mod. Phys. Lett. A **29**, 2655 (1991).
- [3] D. R. Brill and S. Deser, Ann. Phys. (N.Y.) **50**, 548 (1968).
- [4] R. Schoen and S. T. Yau, Phys. Rev. Lett. **43**, 1457 (1979).
- [5] E. Witten, Commun. Math. Phys. **80**, 381 (1981).
- [6] C. C. Liu and S. T. Yau, Phys. Rev. Lett. **90**, 231102 (2003).
- [7] C. M. Chen, J. M. Nester and R. S. Tung, Phys. Lett. A **203**, 5 (1995).
- [8] C. M. Chen and J. M. Nester, Class. Quant. Grav. **16**, 1279 (1999).
- [9] C. C. Chang, J. M. Nester and C. M. Chen, Phys. Rev. Lett. **83**, 1897 (1999).
- [10] C. M. Chen and J. M. Nester, Grav. Cosmol. **6**, 257 (2000).
- [11] C. M. Chen, J. M. Nester and R. S. Tung, Phys. Rev. D **72**, 104020 (2005).
- [12] M. T. Wang and S. T. Yau, Phys. Rev. Lett. **102**, 021101 (2009).
- [13] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993).
- [14] M. Blau and B. Rollier, Class. Quant. Grav. **25**, 105004 (2008).
- [15] P. P. Yu and R. R. Caldwell, Gen. Rel. Grav. **41**, 559 (2009).
- [16] J. M. Nester, L. L. So and T. Vargas, Phys. Rev. D **78**, 044035 (2008). This work includes a discussion of the physical appropriateness of a negative value for the energy of certain cosmological models.
- [17] R. S. Tung, Class. Quant. Grav. **25**, 085005 (2008).